

ON THE MODULAR FUNCTOR ASSOCIATED WITH A FINITE GROUP

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1. INTRODUCTION

In this note, we discuss the complex-algebraic approach to the modular functor constructed from a finite group G . This can be considered as a “baby” case of a more interesting situation, modular functors related to orbifold conformal field theories, which we intend to pursue in a subsequent papers.

We start by recalling some basic facts about the modular functor; detailed exposition can be found, e.g., in [BK]. Let \mathcal{C} be a semisimple abelian category. Then the following structures are essentially equivalent:

- (1) Topological 2-dimensional modular functor, i.e. an assignment

$$(1.1) \quad (\Sigma, p_i, V_i) \mapsto W(\Sigma, p_i, V_i).$$

Here Σ is an oriented 2-dimensional surface with boundary, with marked points p_i on each boundary circle $(\partial\Sigma)_i$ and an object $V_i \in \mathcal{C}$ assigned to $(\partial\Sigma)_i$, and $W(\Sigma, p_i, V_i)$ is a finite-dimensional complex vector space. This assignment should satisfy a number of properties, most important being functoriality and gluing axiom.

- (2) Complex-algebraic modular functor, i.e. a collection of vector bundles with a projectively flat connection $W(C, V_i)$ on the moduli space $\mathcal{M}_{g,n}$ of stable curves C with marked points p_i and non-zero tangent vector $v_i \in T_{p_i}C$. This assignment should satisfy a number of properties, most important being functoriality and factorization properties, which describes behavior of the connection near the boundary of Deligne–Mumford compactification of the moduli space.
- (3) A structure of a modular tensor category on \mathcal{C} .

Note that even though structures of a topological MF and a complex-algebraic MF are equivalent, there is in general no simple way to construct complex-analytic MF from a topological one. This is essentially equivalent to constructing, for a given representation of $\pi_1(\mathcal{M}_{g,n})$, a vector bundle with a flat connection with regular singularities whose monodromy is described by this representation. While the Riemann–Hilbert correspondence shows that such a local system exists, it does not give a natural construction of it.

A simplest example of a modular functor is a MF associated with a finite group G and a cohomology class $\omega \in H^3(G, S^1)$. In topological setting, it arises from the Chern–Simons theory with a finite gauge group G ; a detailed description can be found in [FQ] (we will review it below). In the language of modular categories, the

corresponding modular category is the category of representations of the twisted Drinfeld double $D^\omega(G)$ of the finite group G (see [DPR]).

In this note, we complete the picture by providing a complex-analytic counterpart of the same modular functor. For simplicity, we only describe untwisted case, i.e. $\omega = 1$. A key ingredient of the construction is the moduli space of “admissible G -covers”, introduced in [JKK].

2. DRINFELD DOUBLE OF A FINITE GROUP

In this section, we briefly recall some facts about Drinfeld double of the finite group. Throughout this section, G is a finite group.

For an element $g \in G$, we denote by $\text{Ad}_g: G \rightarrow G$ the adjoint action of g on G : $\text{Ad}_g(x) = gxg^{-1}$. These operators naturally form a groupoid, with the set of objects G and $\text{Mor}(x, y) = \{g \in G \mid \text{Ad}_g(x) = y\}$. We will denote this groupoid by $\text{Ad}(G)$ (this notation is not standard).

An action of the groupoid $\text{Ad}(G)$ on a set X is a decomposition $X = \sqcup_{g \in G} X_g$ and an action of G on X such that $gX_h \subset X_{ghg^{-1}}$. Similarly, a *representation* of the groupoid $\text{Ad}(G)$ is a vector space V with decomposition $V = \bigoplus_{g \in G} V_g$ and a linear action of G such that $gV_h \subset V_{ghg^{-1}}$; in other words, a representation of $\text{Ad}(G)$ is the same as G -equivariant vector bundle on G .

Similar to the usual construction for groups, we can define “group algebra” of the groupoid $\text{Ad}(G)$ as an algebra of formal linear combinations of morphisms in $\text{Ad}(G)$; we define product to be zero if the morphisms are not composable. It is immediate that a representation of $\text{Ad}(G)$ is the same as a representation of the group algebra (as an associative algebra with unit). It is also easy to check that this group algebra is in fact isomorphic to the semidirect product $D(G) = \mathbb{C}[G] \ltimes \mathcal{F}(G)$, where $\mathbb{C}[G]$ is the group algebra of G and $\mathcal{F}(G)$ is the algebra of functions on G . Denoting by $\delta_g \in \mathcal{F}(G)$ the delta function at g , the product in $D(G)$ is given by $g\delta_h = \delta_{\text{Ad}_g(h)}g$. In fact, $D(G)$ has a natural structure of Hopf algebra (see, e.g., [BK]). This Hopf algebra is called *Drinfeld double* of the group G .

As for any Hopf algebra, we can define the notion of invariants: if V is a representation of $D(G)$, then

$$(2.1) \quad V^{D(G)} = \text{Hom}_{D(G)}(\mathbb{C}, V) = (V_1)^G.$$

It is well known that $D(G)$ is a semisimple associative algebra; thus it has finitely many irreducible representations (up to isomorphism). We will denote by $\text{Irr}(D(G))$ the set of isomorphism classes of irreducible representations. Explicit description of these representations can be found, e.g., in [BK]. Semisimplicity also implies that as an algebra,

$$D(G) \simeq \bigoplus_{\lambda \in \text{Irr}(D(G))} \text{End}(\rho_\lambda).$$

3. MODULAR FUNCTOR ASSOCIATED WITH A FINITE GROUP: TOPOLOGICAL DESCRIPTION

In this section, we briefly recall the definition of topological modular functor associated with a finite group G , following [FQ] (with minor changes).

Let X be an oriented compact surface with boundary, and with a marked point $p_i \in (\partial X)_i$ in each boundary component of X . We will call such a structure *marked surface* and will denote $X^m = (X, \{p_i\})$. For such a surface, we denote by

$\mathcal{P}(X^m)$ the category with objects $P^m = (P, \{q_i\})$, where P is a principal G -bundle $\pi: P \rightarrow X$, and $q_i \in \pi^{-1}(p_i)$ is a chosen lifting of the marked points p_i to P . The morphisms in this category are isomorphisms of G -bundles which map marked points to marked points. Note that every such morphism is invertible, so $\mathcal{P}(X^m)$ is a groupoid, and that if each connected component of X^m has at least one boundary component, then objects of $\mathcal{P}(X^m)$ have no non-trivial automorphisms, so $\mathcal{P}(X^m)$ is essentially a set.

For a principal G -bundle with marked points $P^m = (P, \{q_i\})$, we can define *monodromy* of P^m around $(\partial X)_i$ as follows. Orientation of X defines a natural direction of the boundary circle $(\partial X)_i$. Choose a parametrization $\gamma: \mathbb{R}/2\pi\mathbb{Z} \rightarrow (\partial X)_i$ such that $\gamma(0) = p_i$; then there is a unique lifting of γ to a map $\tilde{\gamma}: [0, 2\pi] \rightarrow P$ such that $\tilde{\gamma}(0) = q_i$. We define monodromy $m_i(P^m)$ by

$$(3.1) \quad \gamma(2\pi) = m_i(P^m)\gamma(0).$$

Also, given P^m and an element $g \in G$, we can define a new marked G -bundle by using g to change the marked point q_i :

$$(3.2) \quad \rho_i(g)P^m = (P, \{q_1, \dots, gq_i, \dots, q_n\}).$$

Note that ρ_i is not a morphism in the category $\mathcal{P}(X)$ (in general, $\rho_i(g)P^m$ is not isomorphic to P^m) but a functor $\mathcal{P}(X^m) \rightarrow \mathcal{P}(X^m)$. It is also immediate from direct computation that

$$(3.3) \quad m_i(\rho_i(g)P^m) = g \cdot m_i(P^m) \cdot g^{-1}.$$

We denote by $\overline{\mathcal{P}(X^m)}$ the set of isomorphism classes in $\mathcal{P}(X^m)$. Then one easily sees that m_i and ρ_i descend to $\overline{\mathcal{P}(X^m)}$, giving maps $\overline{\mathcal{P}(X^m)} \rightarrow G$ and action of G on $\overline{\mathcal{P}(X^m)}$. It follows from (3.3) that m_i, ρ_i define an action of the groupoid $\text{Ad}(G)$ on the set $\overline{\mathcal{P}(X^m)}$.

Now, let

$$(3.4) \quad E(X^m) = \mathcal{F}(\overline{\mathcal{P}(X^m)}),$$

where $\mathcal{F}(S)$ stands for the space of functions on S .

Then for each boundary component $(\partial X)_i$, the action of $\text{Ad}(G)$ on $\overline{\mathcal{P}(X^m)}$ by m_i, ρ_i defines on $E(X)$ a structure of representation of $\text{Ad}(G)$ and thus, by results of Section 2, of a representation of the algebra $D(G)$. We will denote this representation by ρ_i . It can be written explicitly as follows:

$$(3.5) \quad \begin{aligned} (\rho_i(\delta_h)f)P^m &= \delta_{h, m_i(P^m)}f(P^m) \\ (\rho_i(g)f)(P^m) &= f(\rho_i(g^{-1})P^m). \end{aligned}$$

Let $\Delta(X)$ be the set of boundary components of X ; for each boundary component $(\partial X)_i, i \in \Delta(X)$, consider a copy $D_i(G)$ of $D(G)$ and let

$$(3.6) \quad D_{\Delta(X)} = \bigotimes_{i \in \Delta(X)} D_i(G).$$

Taking tensor product of actions ρ_i defined by (3.5), we see that $E(X^m)$ has a natural structure of a $D_{\Delta(X)}(G)$ -module.

3.1. Definition. For a marked surface X^m and representations V_i of $D(G)$ assigned to boundary components $(\partial X)_i$, we define the vector space

$$(3.7) \quad W(X^m, \{V_i\}) = \text{Hom}_{D_{\Delta(X)}(G)}(E(X), \bigotimes V_i)$$

It immediately follows from the definition that

$$(3.8) \quad E(X^m) = \bigoplus_{\lambda_1, \dots, \lambda_n} \rho_{\lambda_1}^* \otimes \dots \otimes \rho_{\lambda_n}^* \otimes W(X^m, \rho_{\lambda_1}, \dots, \rho_{\lambda_n})$$

where each λ_i runs over the set $\text{Irr}(D(G))$ of isomorphism classes of irreducible representations of $D(G)$, ρ_{λ_i} is the corresponding representation, and $\rho_{\lambda_i}^*$ is the dual representation.

Finally, we define gluing. Let X^m be a marked surface, and let $c \subset X$ be a simple closed curve (“cut”) with a marked point p . Cutting X along c gives a new surface X_{cut}^m , with $\partial X_{cut} = \partial X \sqcup c' \sqcup c''$ and marked points p', p'' on c', c'' respectively. Let $\mathcal{P}_{c', c''}(X_{cut}^m) \subset \mathcal{P}(X_{cut}^m)$ be the category of marked G -bundles such that $m_{c'}(P)m_{c''}(P) = 1$. We have an action of G by functors on the subcategory $\mathcal{P}_{c', c''}(X_{cut}^m)$ given by $\rho_{c', c''}(g) = \rho_{c'}(g)\rho_{c''}(g)$. Passing to the set of isomorphism classes $\overline{\mathcal{P}(X_{cut}^m)}$, we get a subset

$$\overline{\mathcal{P}_{c', c''}(X_{cut}^m)} \subset \overline{\mathcal{P}(X_{cut}^m)}$$

with an action of G .

Any principal G -bundle on X can be restricted to X_{cut} . Analyzing which bundles on X_{cut} can be obtained in this way and taking into account marked points, one easily gets the following proposition.

3.2. Proposition. *Restriction gives a bijection*

$$\overline{\mathcal{P}(X^m)} \xrightarrow{\sim} \overline{\mathcal{P}_{c', c''}(X_{cut}^m)}/G$$

Passing to functions, we get the following result:

3.3. Theorem. *We have a natural isomorphism*

$$E(X^m) \xrightarrow{\sim} (E(X_{cut}^m))^{D(G)}$$

where the action of $D(G)$ is given by $\rho_{c'} \otimes \rho_{c''}$.

Note that $D(G)$ is not cocommutative, so the action of $D(G)$ on $E(X_{cut}^m)$ depends on the ordering of the cuts c', c'' . However, it is easy to see that the space of invariants does not depend on this choice.

Finally, decomposing $E(X^m)$ into direct sum of irreducibles as in (3.8), we immediately get the following result:

3.4. Theorem. *One has a natural isomorphism of vector spaces*

$$W(X^m, V_1, \dots, V_n) \simeq \bigoplus_{\lambda} W(X_{cut}^m, V_1, \dots, V_n, \rho_{\lambda}, \rho_{\lambda}^*).$$

Now it is easy to check the following result (see [FQ] for details):

3.5. Theorem. *The assignment $X^m, V_1, \dots, V_n \mapsto W(X^m, \{V_i\})$, with the gluing defined in Theorem 3.4, satisfies all axioms of a modular functor. The corresponding modular structure on the category of representations of $D(G)$ coincides with the modular structure defined by the quasi-triangular Hopf algebra structure on $D(G)$.*

4. MODULI SPACE OF G -COVERS

In this section, we briefly recall the definition of the moduli space of stable G -covers, following [JKK], with one modification. Namely, [JKK] describes pointed curves (i.e., curves with marked points p_1, \dots, p_n); we will also require choice of a non-zero tangent vector $v_i \in T_{p_i}C$ at each marked point, or, equivalently, 1-jet of local parameter at p_i . One can easily check that all arguments of [JKK] apply in this case with obvious changes, except for definition of gluing maps, which require more extensive changes and which will be discussed in forthcoming papers.

First, let us recall the usual definition of marked curve and the corresponding moduli space.

4.1. Definition. A *marked curve* is a non-singular complex curve C with distinct marked points p_i and a choice of 1-jet of local parameter dz_i at p_i . A marked curve is *stable* if the group of automorphisms preserving p_i, dz_i is finite.

Note that we do not require that C be connected. For connected curves, stability means that the genus g and number of marked points n are subject to restriction $(g, n) \neq (0, 0), (0, 1), (1, 0)$.

We will denote by $\mathcal{M}_{g,n}$ the moduli stack of connected genus g marked curves with n marked points. It is well-known that $\mathcal{M}_{g,n}$ is a smooth Deligne–Mumford stack.

Now let G be a finite group. The following definition is a minor modification of the one given in [JKK].

4.2. Definition. A non-singular marked G -cover is the following collection of data:

- A non-singular stable marked curve $C, \{p_i\}, \{dz_i\}$
- A finite cover $\pi: \tilde{C} \rightarrow C$ and an action of G on \tilde{C} which preserves projection π and satisfies:
 - Over $C - \{p_1, \dots, p_n\}$, π is a principal G -bundle.
 - Near each $q_i \in \pi^{-1}(p_i)$, the map π is locally analytically equivalent to $\mathbb{C} \rightarrow \mathbb{C}: \tilde{z} \mapsto z = \tilde{z}^{r_i}$ (the number r_i is called the branching index at p_i).
- For each i , a choice of marked point $\tilde{p}_i \in \pi^{-1}(p_i)$ and 1-jet of local parameter $d\tilde{z}_i$ at \tilde{p}_i such that $\tilde{z}^{r_i} = z_i$.

For future use, we note here that the tangent spaces $T_{p_i}C$ and $T_{\tilde{p}_i}\tilde{C}$ are related by

$$(4.1) \quad \begin{aligned} T_{p_i}C &= (T_{\tilde{p}_i}\tilde{C})^{\otimes r_i} = (T_{\pi^{-1}(p_i)}\tilde{C})/G \\ T_{\pi^{-1}(p)}\tilde{C} &= \bigoplus_{q \in \pi^{-1}(p)} T_q\tilde{C}. \end{aligned}$$

As in the topological picture, given a G -cover $\tilde{C} \rightarrow C$ and a marked point $p_i \in C$, one can define the monodromy $m_i(\tilde{C}) \in G$ and action ρ_i of G by $\tilde{p}_i \mapsto g\tilde{p}_i, \tilde{z}_i \mapsto \tilde{z}_i \circ g^{-1}$. Again, trivial check shows that they satisfy relation $m_i(\rho_i(g)\tilde{C}) = g \cdot m_i(\tilde{C}) \cdot g^{-1}$; thus, they define an action of the groupoid $\text{Ad}(G)$ on the category of G -covers.

The notion of G -cover can be easily defined for families of curves (see [JKK]). This allows one to define the moduli space of marked G -covers. We will denote the moduli space of connected genus g marked curves with n marked points by $\mathcal{M}_{g,n}^G$.

The same argument as in [JKK] shows that $\mathcal{M}_{g,n}^G$ is a smooth Deligne–Mumford stack, and the natural forgetting map defined by

$$st(\tilde{C} \rightarrow C) = C$$

gives a morphism of stacks

$$(4.2) \quad st: \mathcal{M}_{g,n}^G \rightarrow \mathcal{M}_{g,n}$$

which is a finite cover.

There is a direct relation of the complex-analytic picture with the topological picture. Let $C, \{p_i\}, \{dz_i\}$ be a marked curve. Choose a local parameter z_i at each p_i with given 1-jet, and a small enough positive real number ε such that z_i is a biholomorphic map $D_i \xrightarrow{\sim} \{z \in \mathbb{C} \mid |z| < \varepsilon\}$, for some neighbourhoods D_i of p_i , such that $D_i \cap D_j = \emptyset$. Let $C^\circ = C \setminus \cup D_i$. This is an oriented surface with boundary. Moreover, we have marked points on the boundary specified by the condition $z_i = \varepsilon$. Thus, C° is a marked surface.

Note that C° depends on the choice of local parameters z_i and ε , but it can be shown that different choices produce marked surfaces which are isomorphic, and isomorphism is canonical up to homotopy (see [BK]). In particular, this implies that the modular functor space $W(C^\circ, V_1, \dots, V_n)$ is canonically defined.

4.3. Theorem. *Let C, C° be as above. Then the category of admissible marked G -covers $\tilde{C} \rightarrow C$ is naturally equivalent to the category $\mathcal{P}(C^\circ)$ of marked G -bundles over C° .*

Proof. First, restrictings a G -cover $\tilde{C} \rightarrow C$ to $C^\circ \subset C$ and forgetting the complex structure defines a functor from G -covers to $\mathcal{P}(C^\circ)$. One easily sees that conversely, given a principal G -bundle on C° , there is a unique way to extend it to a (topological) branched cover \tilde{C} , and then there is a unique complex structure on \tilde{C} compatible with projection $\tilde{C} \rightarrow C$. \square

5. MODULAR FUNCTOR ASSOCIATED WITH A FINITE GROUP: COMPLEX-ALGEBRAIC DESCRIPTION

In this section, we finally formulate the main result of this note, namely a definition of complex-analytic modular functor associated with a finite group G .

Let $\mathcal{M}_{g,n}^G$ be the moduli space of G -covers defined in Section 4. Consider the structure sheaf \mathcal{O} on $\mathcal{M}_{g,n}^G$; it is obviously a module of the sheaf $D_{\mathcal{M}_{g,n}^G}$ of differential operators on $\mathcal{M}_{g,n}^G$. Define a sheaf \mathcal{E} of D -modules on $\mathcal{M}_{g,n}$ by

$$(5.1) \quad \mathcal{E} = st_*(\mathcal{O})$$

where $st: \mathcal{M}_{g,n}^G \rightarrow \mathcal{M}_{g,n}$ is the forgetting map (4.2).

Since $st: \mathcal{M}_{g,n}^G \rightarrow \mathcal{M}_{g,n}$ is a finite cover, one easily sees that \mathcal{E} is a lisse D -module, i.e. a sheaf of sections of some vector bundle E with flat connection. Action of the groupoid $\text{Ad}(G)$ on the moduli space $\mathcal{M}_{g,n}^G$ gives an action of $\text{Ad}(G)$ (and thus, of the algebra $D(G)$) on the sheaf \mathcal{E} . Define $\Delta(C)$ be the set of marked points of C and define, in analogy with (3.7), the modular functor sheaf

$$(5.2) \quad \mathcal{E}(V_1, \dots, V_n) = \text{Hom}_{D_{\Delta(C)}(G)}(\mathcal{E}, \mathcal{O} \otimes V_1 \otimes \dots \otimes V_n).$$

This sheaf is naturally the sheaf of holomorphic sections of the vector bundle E with fiber

$$(5.3) \quad E_C(V_1, \dots, V_n) = \text{Hom}_{D_{\Delta(C)}(G)}(E, V_1 \otimes \dots \otimes V_n).$$

5.1. Theorem. (1) *Let C be a marked curve. Then one has canonical isomorphisms of $D_{\Delta(C)}(G)$ -modules*

$$(5.4) \quad E_C = E(C^\circ)$$

(2) *Let C be a marked curve. Then one has canonical isomorphisms of vector spaces*

$$(5.5) \quad E_C(V_1, \dots, V_n) = W(C^\circ, V_1, \dots, V_n)$$

(3) *Under isomorphism (5.5), the representation of the mapping class groupoid Γ given by monodromy of the local system $E_C(V_1, \dots, V_n)$ is identified with the representation given by the modular functor $W(C^\circ, V_1, \dots, V_n)$.*

In short, this theorem states that under the correspondence between topological MF and complex-analytic MF described in [BK], the modular functor defined by (5.1) corresponds to the finite group modular functor defined in Definition 3.1.

Note that this theorem does not address the question of defining the gluing isomorphism in the complex-analytic approach. This will be discussed in subsequent papers.

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